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Free and free abelian Euler–Satake characteristics of nonorientable 2-orbifolds

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ABSTRACT

We compute the Γ-sectors and Γ-Euler–Satake characteristic of a closed, effective 2-dimensional orbifold Q where Γ is a free or free abelian group. Using this information, we determine a family of orbifolds such that the complete collection of Γ-Euler–Satake characteristics associated to free and free abelian groups determines the number and type of singular points of Q as well as the Euler characteristic of the underlying space. Additionally, we show that any collection of these groups whose Euler–Satake characteristics determine this information contains both free and free abelian groups of arbitrarily large rank. It follows that the collection of Euler–Satake characteristics associated to free and free abelian groups constitute a finer orbifold invariant than the collection of Euler–Satake characteristics associated to free groups or free abelian groups alone.

1. Introduction

In [21] and [22], Tamanoi introduced a large number of invariants for *global quotient orbifolds*, orbifolds given by the quotient of a manifold by a finite group. These invariants are defined by applying existing invariants to sector decompositions of the orbifold. In [8] and [9], the second author and Carla Farsi generalized the construction of these sector decompositions to the case of a general orbifold Q, defining for each finitely generated discrete group Γ the *Γ-sectors of Q*, a disjoint union of orbifolds denoted by $\tilde{Q}_\Gamma$. This construction generalizes the inertia orbifold, which coincides with the case $\Gamma = \mathbb{Z}$.

As the orbifold of Γ-sectors of Q is an orbifold invariant for each Γ, it follows that any invariant for orbifolds applied to $\tilde{Q}_\Gamma$ determines a new invariant of Q. For instance, if we let $\chi_{ES}(Q)$ denote the Euler–Satake characteristic of Q defined in [19], we can define for each Γ the Γ-Euler–Satake characteristic $\chi_\Gamma^{ES}(Q) = \chi_{ES}(\tilde{Q}_\Gamma)$, which is finite when Q is compact and Γ is finitely generated. For different choices of Γ, this definition recovers the stringy orbifold Euler characteristic defined in [6] for global quotients and [17] for general orbifolds, and the generalized orbifold Euler characteristics given for global quotient orbifolds in [3,21], and [22]. See [9] for the relationship between these invariants and their resulting generalizations to orbifolds that are not presented as global quotients.

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Given such a large class of invariants for orbifolds, it is natural to ask whether these invariants give any new information about the orbifold. Is it the case that every group Γ introduces new invariants for orbifolds? Or are these invariants rather determined by those associated to some smaller class of groups?

In order to explore this question, the authors and Whitney DuVal determined the degree to which the collection of Γ-Euler–Satake characteristics classify closed, effective, 2-orbifolds in [7]. It was found that the Euler–Satake characteristics associated to free abelian groups $\mathbb{Z}^l$ do classify these orbifolds, and moreover that groups of arbitrarily large rank are required to do so. The free abelian Euler–Satake characteristics contain the Euler characteristics of the underlying topological spaces of the $\mathbb{Z}^l$-sectors, which are as usual determined by the (additive) cohomology, (additive rational) K-theory, etc. Therefore, these invariants applied to the $\mathbb{Z}^l$-sectors do yield new invariants for orbifolds. Note that all orientable 2-orbifolds Q have abelian local groups, and hence if $\mathbb{F}_l$ denotes the free group with l generators, then $\chi^{ES}_{\mathbb{Z}^l}(Q) = \chi^{ES}_{\mathbb{F}_l}(Q)$. It follows that the free and free abelian Euler–Satake characteristics contain exactly the same information about orientable 2-orbifolds.

Here, we turn our attention to the nonorientable 2-orbifolds to determine whether the collection of Euler–Satake characteristics associated to free and free abelian groups contain more information than the Euler–Satake characteristics associated to free groups alone or free abelian groups alone. Note that it was demonstrated in [7] that nonorientable 2-orbifolds cannot be classified by Γ-Euler–Satake characteristics for any collection of groups Γ. However, we show the following, which demonstrates that the collection of Euler–Satake characteristics associated to these groups determines a great deal about the singular set of the orbifold.

Theorem 1.1. *Suppose Q and Q' are closed, connected, effective 2-orbifolds that do not have cone points of order 4 such that $\chi^{ES}_{\mathbb{Z}^l}(Q) = \chi^{ES}_{\mathbb{Z}^l}(Q')$ for infinitely many positive integers l and $\chi^{ES}_{\mathbb{F}_\lambda}(Q) = \chi^{ES}_{\mathbb{F}_\lambda}(Q')$ for infinitely many positive integers λ. Then $\chi_{top}(Q) = \chi_{top}(Q')$, and Q and Q' have the same number of cone points and corner reflectors of each order.*

We also obtain the following, which is a consequence of Corollaries 3.4 and 4.4 below.

Corollary 1.1. *Suppose Q and Q' are closed, connected, effective 2-orbifolds. Then $\chi^{ES}_{\Gamma}(Q) = \chi^{ES}_{\Gamma}(Q')$ for every finitely generated discrete group Γ if and only if $\chi_{top}(Q) = \chi_{top}(Q')$ and Q and Q' have the same number of cone points and corner reflectors of each order.*

It follows that no collection of Γ-Euler–Satake characteristics can distinguish between orbifolds Q and Q' that satisfy the consequence of Theorem 1.1. Hence, Theorem 1.1 cannot be improved upon by considering larger classes of groups Γ. Moreover, we also demonstrate the following.

Theorem 1.2. *Let $L \geqslant 0$ be an integer.*

I. *There exist closed, connected, effective 2-orbifolds Q and Q' that have no cone points of order 4 such that*
   i. *Q and Q' have no cone points nor corner reflectors of the same order,*
   ii. *$\chi^{ES}_{\mathbb{F}_\lambda}(Q) = \chi^{ES}_{\mathbb{F}_\lambda}(Q')$ for each nonnegative integer λ, and*
   iii. *$\chi^{ES}_{\mathbb{Z}^l}(Q) = \chi^{ES}_{\mathbb{Z}^l}(Q')$ for each $l \leqslant L$.*

II. *There exist closed, connected, effective 2-orbifolds $\mathcal{Q}$ and $\mathcal{Q}'$ that have no cone points of order 4 such that*
   i. *$\mathcal{Q}$ and $\mathcal{Q}'$ have no cone points nor corner reflectors of the same order,*
   ii. *$\chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}) = \chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}')$ for each nonnegative integer l, and*
   iii. *$\chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}) = \chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}')$ for each $\lambda \leqslant L$.*

It follows that any collection of free or free abelian groups that classify closed, connected, effective 2-orbifolds without cone points of order 4 as well as Theorem 1.1, i.e. as well as possible, must contain both free groups of arbitrarily large rank and free abelian groups of arbitrarily large rank. Therefore, Theorem 1.1 cannot be improved upon by considering any smaller collection of free or free abelian groups, and moreover, a collection of Γ-Euler–Satake characteristics corresponding to both free and free abelian groups of arbitrarily large rank is a finer invariant than any collection with either finitely many free or finitely many free abelian groups.

The rather irritating hypothesis that the orbifolds in question do not have cone points of order 4 is in fact necessary, as is demonstrated by Example 4.1. Hence, if one asks whether the Γ-Euler–Satake characteristics associated to free and free abelian groups contain more information than a subcollection with finitely many free or finitely many free abelian groups, an appropriate class of orbifolds with which to answer this question in the affirmative is that of closed, connected, effective 2-orbifolds without cone points of order 4. If, on the other hand, one is interested in a collection of Γ-Euler–Satake characteristics that determine as much information as possible about closed, connected, effective 2-orbifolds, a much more natural class to consider, then we demonstrate that any collection of Γ containing infinitely many free groups, infinitely many free abelian groups, and $\mathbb{Z}/2\mathbb{Z}$ is sufficient. This alternative perspective is developed in Section 4.3.

This paper is organized in four sections as follows. In Section 2, we collect the necessary background information on orbifolds, Γ-sectors, and the Euler–Satake characteristic. In Section 3, we explicitly determine the Γ-sectors of closed, effective 2-orbifolds where Γ is free or free abelian, as well as the Γ-extensions of invariants associated to these groups. The description of the sectors is developed in Section 3.1 and stated as Lemmas 3.1 and 3.2. In Section 3.2, we use this description to compute the associated Γ-Euler–Satake characteristics of these orbifolds, as well as the associated extensions of the Euler characteristic and Betti numbers. Section 4 is devoted to the proofs of the main results above. In particular, we prove Proposition 4.2, which implies Theorem 1.1, in Section 4.1. We prove Theorem 1.2 in Section 4.2. Section 4.3 extends these results to determine a collection of groups Γ such that the corresponding Euler–Satake characteristics completely determine the singularities of closed, effective 2-orbifolds.

2. Preliminaries

In this section, we give the appropriate background information on orbifolds and Γ-sectors. We adopt the perspective introduced in [14] and [15], considering orbifolds to be Morita equivalence classes of proper, étale, Lie groupoids; following [1], we do not require orbifolds to be effective.

An orbifold Q is given by a second-countable Hausdorff topological space $\mathbb{X}_Q$ along with a homeomorphism between $\mathbb{X}_Q$ and the orbit space of a proper, étale Lie groupoid $\mathcal{G}$. Given such a presentation, we identify $\mathbb{X}_Q$ with the orbit space $|\mathcal{G}|$ of $\mathcal{G}$ and avoid specific reference to the homeomorphism. We refer to $\mathbb{X}_Q$ as the *underlying space* of the orbifold Q and $\mathcal{G}$ as a *presentation* of Q. We say that Q is *connected* when $\mathbb{X}_Q$ is connected and Q is *closed* when $\mathbb{X}_Q$ is compact; note that we do not consider orbifolds with boundary. We let G_0 and G_1 denote the manifolds of objects and arrows, respectively, of the groupoid $\mathcal{G}$. If $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent groupoids, then their orbit spaces are homeomorphic, and they present the same orbifold structure on this space. Two orbifolds are said to be *diffeomorphic* if they are presented by Morita equivalent groupoids. A point $p \in \mathbb{X}_Q$ is *nonsingular* if it represents the orbit of a point in G_0 with trivial isotropy and *singular* otherwise. As the isotropy groups of points in the same orbit are isomorphic, these notions are well-defined.

Let Γ be a finitely generated discrete group, which can be treated as a groupoid with a single object. Then a groupoid homomorphism $\phi_x : \Gamma \to \mathcal{G}$ is precisely a choice of a point $x \in G_0$ and a group homomorphism $\Gamma \to G_x$, also denoted by ϕ_x, into the isotropy group at x. Hence, as a set, $HOM(\Gamma, \mathcal{G}) = \coprod_{x \in G_0} HOM(\Gamma, G_x)$. This set inherits the structure of a disjoint union of smooth manifolds from $\mathcal{G}$. As well, there is a natural action of $\mathcal{G}$ on $HOM(\Gamma, \mathcal{G})$ with anchor map $\phi_x \to x$. That is, if $\phi_x \in HOM(\Gamma, \mathcal{G})$ and $g \in G_1$ with source x, then we define $g\phi_x \in HOM(\Gamma, \mathcal{G})$ by

$$(g\phi_x)(\gamma) = g\phi_x(\gamma)g^{-1}, \quad \forall \gamma \in \Gamma.$$

The resulting translation groupoid $\mathcal{G}^\Gamma = \mathcal{G} \ltimes HOM(\Gamma, \mathcal{G})$ is a proper, étale, Lie groupoid, and hence presents an orbifold denoted by $\tilde{Q}_\Gamma$, called the *orbifold of Γ-sectors of* Q. In general, $\tilde{Q}_\Gamma$ is disconnected with connected components of varying dimension. If $\mathcal{G}$ and $\mathcal{G}'$ are Morita equivalent, then $\mathcal{G}^\Gamma$ and $(\mathcal{G}')^\Gamma$ are Morita equivalent as well, so that the orbifold $\tilde{Q}_\Gamma$ is well-defined. If $\phi_x, \phi_y \in HOM(\Gamma, \mathcal{G})$, we say that $\phi_x \approx \phi_y$ if they represent orbits in the same connected component of $\tilde{Q}_\Gamma$. We let (ϕ_x) denote the $\approx$-class of ϕ_x and let $\tilde{Q}_{(\phi_x)}$ denote the connected orbifold associated to this $\approx$-class. Intuitively, $\phi_x \approx \phi_y$ indicates that x and y represent orbits in the same singular stratum of $\mathbb{X}_Q$, and the homomorphisms ϕ_x and ϕ_y coincide via the identifications of the local structure of Q at x and y. In particular, if Q is connected, and ϕ_x and ϕ_y are trivial homomorphisms, then $\phi_x \approx \phi_y$. The corresponding connected component of $\tilde{Q}_\Gamma$, denoted by $\tilde{Q}_{(1)}$, is diffeomorphic to Q. See [8, Section 2] for more details.

In the case that Q is a global quotient orbifold, i.e. Q is presented by $G \ltimes M$ where M is a smooth manifold and G is a finite group of diffeomorphisms, we have that $\tilde{Q}_\Gamma$ is presented by

$$\coprod_{[\phi] \in HOM(\Gamma, G)/G} C_G(\phi) \ltimes M^{\langle \phi \rangle}.$$

Here, G acts on $HOM(\Gamma, G)$ by pointwise conjugation and $[\phi]$ denotes the conjugacy class of a homomorphism ϕ, $M^{\langle \phi \rangle}$ denotes the points fixed by the image of ϕ, and $C_G(\phi)$ is the centralizer of the image of ϕ in G. This definition was originally given by Tamanoi in [21] and [22]. That this orbifold coincides with $\mathcal{G}^\Gamma$ for another presentation $\mathcal{G}$ of Q is demonstrated in [9].

For the reader more familiar with orbifold structures defined by an atlas (as in [18,19,23,2,4]), we note that every orbifold is locally a global quotient as above, and the construction of the Γ-sectors for global quotients extends to these charts. Consider an orbifold chart of the form $\{V, G, \pi\}$ where V is an open subset of $\mathbb{R}^n$, G is a finite subgroup of $O(n)$, and $\pi : V \to \mathbb{X}_Q$ induces a homeomorphism between V/G and an open subset of $\mathbb{X}_Q$. Then each $\phi : \Gamma \to G$ corresponds to a Γ-sector locally modeled by the $C_G(\phi)$-action on $V^{\langle \phi \rangle}$. Injections of charts for Q restrict in the obvious way to injections of charts for $\tilde{Q}_\Gamma$, patching these local descriptions together to form the orbifold of Γ-sectors. See [5] or [12] for the construction of the inertia orbifold, which corresponds to the case $\Gamma = \mathbb{Z}$, from this perspective.

The *Euler–Satake characteristic* $\chi_{ES}(Q)$ of a closed orbifold Q was defined in [19] in terms of a simplicial decomposition $\mathcal{T}$ of $\mathbb{X}_Q$ such that for each simplex $\sigma \in \mathcal{T}$, the order of the isotropy group is constant on the interior of σ. Letting $|G_\sigma|$ denote this order, we have

$$\chi_{ES}(Q) = \sum_{\sigma \in \mathcal{T}} \frac{1}{|G_\sigma|}(-1)^{\dim \sigma} \in \mathbb{Q}.$$

If Q is a global quotient presented by $G \ltimes M$, then $\chi_{ES}(Q) = \chi(M)/|G|$. We use $\chi_{top}(Q)$ to denote $\chi(\mathbb{X}_Q)$, the usual Euler characteristic of the underlying space. We let $b_i(Q)$ denote the ith Betti number of $\mathbb{X}_Q$. Note that by [1, Theorem 2.13], the Betti numbers of the underlying space of Q coincide with the Betti numbers defined in terms of the ranks of the de Rham cohomology of Q, in the orbifold sense, so this latter definition offers no confusion.

For a finitely generated discrete group Γ, we let χ_Γ^{ES}, χ_Γ^{top}, and b_Γ^i denote the Γ-extensions of the invariants χ_{ES}, χ_{top}, and b_i, respectively, in the sense of [22] and [11]; that is, we set

$$\chi_\Gamma^{ES}(Q) = \chi_{ES}(\tilde{Q}_\Gamma),$$
$$\chi_\Gamma^{top}(Q) = \chi_{top}(\tilde{Q}_\Gamma),$$

and

$$b_\Gamma^i(Q) = b_i(\tilde{Q}_\Gamma).$$

Note that the Γ-sectors are defined for an arbitrary discrete group Γ, though they may have an infinite number of connected components when Γ is not finitely generated or Q is not compact. Hence, we restrict to the case of Q closed and Γ finitely generated to ensure that the above invariants are finite.

We are primarily interested in the case of *effective* orbifolds, i.e. orbifolds presented by effective groupoids or equivalently orbifolds locally modeled by the quotient of an effective group action. However, the Γ-sectors of such an orbifold need not be effective. For emphasis, we use the notation $G \ltimes_{triv} M$ for the translation groupoid given by a trivial G-action on M.

If Q is a closed, effective 2-dimensional orbifold, in particular without boundary as an orbifold, then $\mathbb{X}_Q$ is a compact surface, possibly with boundary as a topological manifold. To avoid confusion, we will always refer to the boundary of $\mathbb{X}_Q$ as the (*manifold*) *boundary* of the underlying space of Q. The singular points of Q that do not lie on the (manifold) boundary of the underlying space are necessarily isolated and locally modeled by $\mathbb{Z}/n\mathbb{Z} \ltimes \mathbb{R}^2$ where $\mathbb{Z}/n\mathbb{Z}$ acts as rotations. These points are called *cone points of order n*. Points on the (manifold) boundary of $\mathbb{X}_Q$ are necessarily singular. Finitely many points on each (manifold) boundary component have dihedral isotropy and are locally modeled by the standard action of the dihedral group D_{2n} with $2n$ elements on $\mathbb{R}^2$; these points are called *corner reflectors of order n*. The remaining points have $\mathbb{Z}/2\mathbb{Z}$-isotropy, and each line segment consisting of these points is referred to as a *reflector line*. Following [16], we refer to a (manifold) boundary component with corner reflectors as a *crown*. See [23] or [2] for more details.

3. Invariants of 2-orbifolds associated to free and free abelian groups

3.1. *The sectors of an effective* 2*-orbifold*

In this subsection, we determine the components of the orbifold of Γ-sectors of a 2-orbifold when Γ is a finitely generated free or free abelian group. This is given in Lemmas 3.1 and 3.2 below. First, we recall some elementary facts about the standard action of a dihedral group on $\mathbb{R}^2$.

Let D_{2n} be the dihedral group of order $2n$. Then D_{2n} acts on $V = \mathbb{R}^2$ in the usual way. That is, $D_{2n} = \langle a, b\colon\ a^n = b^2 = 1, ab = ba^{n-1}\rangle$ where a acts via a rotation through the angle $2\pi/n$ and b acts via reflection through a fixed line through the origin in V. If n is odd, then the conjugacy classes of D_{2n} are

$$\{1\};\ \{a, a^{n-1}\};\ \{a^2, a^{n-2}\};\ \dots;\ \{a^{(n-1)/2}, a^{(n+1)/2}\};\ \{b, ab, a^2b, \dots, a^{n-1}b\}.$$

In particular, the n reflections are all conjugate. It follows that if $H = \langle a^q\rangle$ is a subgroup of nontrivial rotations, then $C_{D_{2n}}(H) \ltimes V^H$ corresponds to a single point with trivial $\langle a\rangle = \mathbb{Z}/n\mathbb{Z}$-action. If $H = \langle a^q b\rangle$ is a subgroup generated by a single reflection, then $C_{D_{2n}}(H) \ltimes V^H$ corresponds to a line with trivial $\langle a^q b\rangle = \mathbb{Z}/2\mathbb{Z}$-action. If H contains any nontrivial rotation and any reflection, then $C_{D_{2n}}(H) \ltimes V^H$ consists of a point with the action of the trivial group.

If n is even, then $a^{n/2}$ is a nontrivial central element. The conjugacy classes are

$$\{1\};\ \{a, a^{n-1}\};\ \dots;\ \{a^{(n-2)/2}, a^{(n+2)/2}\};\ \{a^{n/2}\};\ \{b, a^2b, \dots, a^{n-2}b\};\ \{ab, a^3b, \dots, a^{n-1}b\}.$$

It follows that if $H = \langle a^{n/2}\rangle$, then $C_{D_{2n}}(H) \ltimes V^H$ is a point with trivial D_{2n}-action. If $H = \langle a^q\rangle$ is a subgroup of nontrivial rotations with $q \neq n/2$, then $C_{D_{2n}}(H) \ltimes V^H$ corresponds to a single point with trivial $\langle a\rangle = \mathbb{Z}/n\mathbb{Z}$-action. If $H = \langle a^q b\rangle$ is a subgroup generated by a single reflection or $H = \langle a^{n/2}, a^q b\rangle$, then $C_{D_{2n}}(H) \ltimes V^H$ corresponds to a line or point with $\langle a^{n/2}, a^q b\rangle = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action where the first summand acts trivially and the second acts via $x \mapsto -x$. If H contains any nontrivial rotation a^q with $q \neq n/2$ and any reflection, then $C_{D_{2n}}(H) \ltimes V^H$ consists of a point with trivial $\langle a^{n/2}\rangle = \mathbb{Z}/2\mathbb{Z}$-action.

With this, we are prepared to determine the free and free abelian sectors of Q. Note that $\mathbb{I}$ denotes the *mirrored interval*, the 1-dimensional orbifold given by a line segment with $\mathbb{Z}/2\mathbb{Z}$-isotropy at the endpoints.

Lemma 3.1. *Let Q be a closed, effective* 2*-orbifold with k cone points of orders $m_1, m_2, \dots, m_k$, o corner reflectors of odd orders $n_1, n_2, \dots, n_o$, and e corner reflectors of even orders $n_{o+1}, n_{o+2}, \dots, n_{o+e}$. We let c denote the number of connected components of the* (*manifold*) *boundary of the underlying space of Q and let d denote the number of* (*manifold*) *boundary components that contain no even corner reflectors. For an integer $l \geqslant 0$, the orbifold $\tilde{Q}_{\mathbb{F}_l}$ of $\mathbb{F}_l$-sectors consists of the following components*:

- *one sector $\tilde{Q}_{(1)}$ diffeomorphic to Q,*
- *$d(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$, circles with trivial $\mathbb{Z}/2\mathbb{Z}$-action,*
- *$e(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$, mirrored intervals with trivial $\mathbb{Z}/2\mathbb{Z}$-action and isotropy groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ at the endpoints,*
- *for each $i = 1, \dots, k$,*
  - *$m_i^l - 1$ sectors diffeomorphic to $\mathbb{Z}/m_i\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/m_i\mathbb{Z}$-action,*
- *for each $j = 1, \dots, o$,*
  - *$(n_j^l - 1)/2$ sectors diffeomorphic to $\mathbb{Z}/n_j\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/n_j\mathbb{Z}$-action,*
  - *$(n_j^{l-1} - 1)(2^l - 1)/2$ sectors diffeomorphic to points with trivial isotropy,*
- *for each $j = o + 1, \dots, o + e$,*
  - *$2^l - 1$ sectors diffeomorphic to $D_{2n_j} \ltimes_{triv} \{pt\}$, points with trivial D_{2n_j}-action,*
  - *$(n_j^l - 2^l)/2$ sectors diffeomorphic to $\mathbb{Z}/n_j\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/n_j\mathbb{Z}$-action,*
  - *$4^l - 3 \cdot 2^l + 2$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action, and*
  - *$(n_j^{l-1} - 2^{l-1})(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/2\mathbb{Z}$-action.*

Proof. Let $\mathcal{G}$ be an orbifold groupoid representing Q, and let $\phi_x : \mathbb{F}_l \to G_x$ be a homomorphism into the isotropy group G_x of some point $x \in G_0$. Then ϕ_x is determined by the image of each of the l generators of $\mathbb{F}_l$. If the image $\operatorname{Im}\phi_x$ of ϕ_x is the trivial group, then $\tilde{Q}_{(\phi_x)} = \tilde{Q}_{(1)}$ is a sector diffeomorphic to Q; see [8]. Suppose x is a point on a reflector line. Then the orbit of x lies on a (manifold) boundary component C of $\mathbb{X}_Q$, which is necessarily homeomorphic to S^1. We first consider the case that C contains no even-order corner reflectors. If C contains no corner reflectors, then every point in C lies on the same reflector line, and hence the $\tilde{Q}_{(\phi_x)}$ is given by $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$. If, on the other hand, C contains odd-order corner reflectors, then let y be a corner reflector of order n_j odd, locally modeled by $D_{2n_j} \ltimes \mathbb{R}^2$. As every reflection in D_{2n_j} is conjugate, every point representing a reflector line in $D_{2n_j} \ltimes \mathbb{R}^2$ has an isotropy group in the same D_{2n_j}-conjugacy class. It follows that for each $z \in G_0$ representing a point on a reflector line in C, there is a homomorphism $\phi_z : \mathbb{F}_l \to G_z$ such that $\phi_x \approx \phi_z$. Applying this argument to each corner reflector on C, we see that $\tilde{Q}_{(\phi_x)}$ is given by $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$, and moreover that every homomorphism ψ_w into the isotropy group of a point w whose orbit is on a reflector line in C is equivalent to a homomorphism into G_x. It follows that the sectors associated to reflector lines in C are determined by choices of nontrivial homomorphisms $\mathbb{F}_l \to G_x = \mathbb{Z}/2\mathbb{Z}$, of which there are $2^l - 1$ possibilities. As there are d such choices for C, there are $d(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$.

Now suppose C is a crown that contains r even-order corner reflectors. If $w \in G_0$ represents such a corner reflector of even order n_j, then there are two conjugacy classes of reflections in $G_w = D_{2n_j}$, corresponding to fixed points that represent orbits on either side of the corner reflector in a neighborhood in C. It follows that $\tilde{Q}_{(\phi_x)}$ is diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$, a noneffective mirrored interval with $\mathbb{Z}/2\mathbb{Z}$-isotropy on the interior and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-isotropy at the endpoints. The endpoints of $\tilde{Q}_{(\phi_x)}$ correspond to adjacent even-order corner reflectors if C contains more than one even-order corner reflector, or to the same corner reflector in the case that C contains only one even-order corner reflector. Noting that C is divided into r segments by the even-order corner reflectors, it follows that each such sector corresponding to points in C is determined by one of the r segments on C and a homomorphism into the isotropy group $\mathbb{Z}/2\mathbb{Z}$ of the interior of the segment, of which there are $r(2^l - 1)$ choices. Considering all crowns with even-order corner reflectors, there are $e(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$.

If x is a cone point, then $G_x = \mathbb{Z}/m_i\mathbb{Z}$ for some i, and hence $|HOM(\mathbb{F}_l, \mathbb{Z}/m_i\mathbb{Z})| - 1 = m_i^l - 1$. Each of these homomorphisms corresponds to a sector given by a single point with trivial $\mathbb{Z}/m_i\mathbb{Z}$-action.

We now consider the case that x is a corner reflector of odd order so that $G_x = D_{2n_j}$ with n_j odd. We let $V_x \subseteq G_0$ be an open subset diffeomorphic to $\mathbb{R}^2$ such that $\mathcal{G}|_{V_x} = G_x \ltimes V_x$. If the image $\operatorname{Im}\phi_x$ is the group generated by a single reflection in D_{2n_j}, then ϕ_x is equivalent to a homomorphism into the isotropy group of a reflector line, and hence the corresponding sector has already been considered. Counting the images of the l generators of $\mathbb{F}_l$, there are $(2^l - 1)n_j$ such homomorphisms. If $\operatorname{Im}\phi_x$ is a nontrivial group of rotations in D_{2n_j}, then $C_{G_x}(\phi_x) \ltimes V_x^{\langle\phi_x\rangle} = \mathbb{Z}/n_j\mathbb{Z} \ltimes \{pt\}$. Counting the images of the generators in this case, there are $n_j^l - 1$ such homomorphisms, and hence as each is contained in a conjugacy class of size two, there are $(n_j^l - 1)/2$ such sectors. The remaining $(n_j^l - n_j)(2^l - 1)$ nontrivial elements of $HOM(\mathbb{F}_l, G_x)$ are the homomorphisms with nonabelian image, containing at least one reflection and nontrivial rotation. Each fixes a point and has trivial centralizer. Hence, they correspond to $(n_j^l - n_j)(2^l - 1)/(2n_j) = (n_j^{l-1} - 1)(2^l - 1)/2$ sectors given by a point with trivial isotropy.

If x is a corner reflector of even order, we again let $G_x = D_{2n_j}$ with n_j even, and let D_{2n_j} have generators a and b as above. The $2^l - 1$ elements of $HOM(\mathbb{F}_l, G_x)$ with image $\langle a^{n_j/2}\rangle$ each define a singleton conjugacy class, and hence a sector given by $D_{2n_j} \ltimes_{triv} \{pt\}$. The $n_j^l - 2^l$ homomorphisms whose image is a nontrivial subgroup of $\langle a\rangle$ that is not $\langle a^{n_j/2}\rangle$ have conjugacy classes of size two, and hence define $(n_j^l - 2^l)/2$ sectors diffeomorphic to $\mathbb{Z}/n_j\mathbb{Z} \ltimes_{triv} \{pt\}$. As in the case of n_j odd, the $(2^l - 1)n_j$ homomorphisms whose image is generated by a single reflection correspond to sectors given by $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$ already considered. Of the remaining nontrivial homomorphisms, there are $(4^l - 3 \cdot 2^l + 2)n_j/2$ whose image is abelian. This can be seen by noting that there are $(3^l - 2^{l+1} + 1)n_j$ homomorphisms that map the generators of $\mathbb{F}_l$ to a single reflection, $a^{n_j/2}$, and the identity; and there are $(4^l - 2 \cdot 3^l + 2^l)n_j/2$ homomorphisms that map generators of $\mathbb{F}_l$ to a single reflection, $a^{n_j/2}$, the product of the reflection and $a^{n_j/2}$, and the identity. Each of these is centralized by the group isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by the single reflection and $a^{n_j/2}$, and hence they define $4^l - 3 \cdot 2^l + 2$ sectors given by $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$. The remaining $(n_j^l - 2^{l-1}n_j)(2^l - 1)$ nontrivial homomorphisms have nonabelian image and conjugacy classes of size n_j, and hence define $(n_j^{l-1} - 2^{l-1})(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$. □

Lemma 3.2. *Let Q be a closed, effective 2-orbifold with singularities denoted as in Lemma 3.1. For an integer $l \geqslant 0$, the orbifold $\tilde{Q}_{\mathbb{Z}^l}$ of $\mathbb{Z}^l$-sectors consists of the following components*:

- *one sector $\tilde{Q}_{(1)}$ diffeomorphic to Q,*
- *$d(2^l - 1)$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$, circles with trivial $\mathbb{Z}/2\mathbb{Z}$-action,*
- *$e(2^l - 1)$ sectors diffeomorphic $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$, mirrored intervals with trivial $\mathbb{Z}/2\mathbb{Z}$-action and isotropy groups $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ at the endpoints,*
- *for each $i = 1, \dots, k$,*
  - *$m_i^l - 1$ sectors diffeomorphic to $\mathbb{Z}/m_i\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/m_i\mathbb{Z}$-action,*
- *for each $j = 1, \dots, o$,*
  - *$(n_j^l - 1)/2$ sectors diffeomorphic to $\mathbb{Z}/n_j\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/n_j\mathbb{Z}$-action,*
- *for each $j = o + 1, \dots, o + e$,*
  - *$2^l - 1$ sectors diffeomorphic to $D_{2n_j} \ltimes_{triv} \{pt\}$, points with trivial D_{2n_j}-action,*
  - *$(n_j^l - 2^l)/2$ sectors diffeomorphic to $\mathbb{Z}/n_j\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/n_j\mathbb{Z}$-action, and*
  - *$4^l - 3 \cdot 2^l + 2$ sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$, points with trivial $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$-action.*

Proof. The $\mathbb{Z}^l$-sectors of Q are exactly the $\mathbb{F}_l$-sectors corresponding to homomorphisms with abelian image. Hence we need only exclude the homomorphisms with nonabelian image, which were noted in the proof of Lemma 3.1. □

Note that the hypothesis that Q is compact in Lemmas 3.1 and 3.2 is required only in order to ensure that Q has finite numbers of cone points and corner reflectors, and that the (manifold) boundary of the underlying space of Q is compact. If Q is a non-compact, effective 2-orbifold without boundary (as an orbifold), then a similar argument can be used to determine the number and diffeomorphism-class of sectors associated to each stratum of the singular set. The only distinction to note is that components of the (manifold) boundary of the underlying space need not be compact, and hence sectors with underlying space homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$ may occur.

3.2. Free and free abelian extensions of invariants

In this subsection, we compute formulas for the Γ-Euler–Satake characteristic, Γ-Euler characteristic, and Γ-Betti numbers of closed, effective 2-orbifolds when Γ is a free or free abelian group. Given Lemmas 3.1 and 3.2, these computations are straightforward; hence, we only give the details for the Γ-Euler–Satake characteristics.

Corollary 3.3 *(Γ-Euler–Satake characteristics). Let Q be a closed, effective 2-orbifold with singularities denoted as in Lemma 3.1. Then for each integer $l \geqslant 0$,*

$$\chi^{ES}_{\mathbb{F}_l}(Q) = \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1} + 2^{l-1} \sum_{j=1}^{o+e} \left(n_j^{l-1} - 1\right), \tag{1}$$

and

$$\chi^{ES}_{\mathbb{Z}^l}(Q) = \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1} + \frac{1}{2} \sum_{j=1}^{o} \left(n_j^{l-1} - 1\right) + \frac{1}{2} \sum_{j=o+1}^{o+e} \left[2^{l-1}\left(2^l - 3\right) + n_j^{l-1}\right]. \tag{2}$$

Proof. By [23, Eq. (13.3.4)], we have that

$$\chi_{ES}(\tilde{Q}_{(1)}) = \chi_{ES}(Q) = \chi_{top}(Q) - \sum_{i=1}^{k} (1 - 1/m_i) - \frac{1}{2} \sum_{j=1}^{o+e} (1 - 1/n_j).$$

The sectors diffeomorphic to $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} S^1$ and $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \mathbb{I}$ all have Euler–Satake characteristic zero. The remaining sectors all consist of points equipped with the trivial action of a finite group, and $\chi_{ES}(H \ltimes_{triv} \{pt\}) = 1/|H|$. Therefore, we have that $\chi^{ES}_{\mathbb{F}_l}(Q)$ is given by

$$\chi_{top}(Q) - \sum_{i=1}^{k}(1-1/m_i) - \frac{1}{2}\sum_{j=1}^{o+e}(1-1/n_j) + \sum_{i=1}^{k}\bigl(m_i^l - 1\bigr)/m_i$$
$$+ \sum_{j=1}^{o}\bigl[\bigl(n_j^l - 1\bigr)/(2n_j) + \bigl(n_j^{l-1} - 1\bigr)\bigl(2^l - 1\bigr)/2\bigr]$$
$$+ \sum_{j=o+1}^{o+e}\bigl[\bigl(2^l - 1\bigr)/(2n_j) + \bigl(n_j^l - 2^l\bigr)/(2n_j) + \bigl(4^l - 3\cdot 2^l + 2\bigr)/4 + \bigl(n_j^{l-1} - 2^{l-1}\bigr)\bigl(2^l - 1\bigr)/2\bigr]$$
$$= \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1} + 2^{l-1}\sum_{i=1}^{o+e}\bigl(n_j^{l-1} - 1\bigr).$$

Similarly,

$$\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{top}(Q) - \sum_{i=1}^{k}(1-1/m_i) - \frac{1}{2}\sum_{j=1}^{o+e}(1-1/n_j) + \sum_{i=1}^{k}\bigl(m_i^l - 1\bigr)/m_i + \sum_{j=1}^{o}\bigl(n_j^l - 1\bigr)/(2n_j)$$
$$+ \sum_{j=o+1}^{o+e}\bigl[\bigl(2^l - 1\bigr)/(2n_j) + \bigl(n_j^l - 2^l\bigr)/(2n_j) + \bigl(4^l - 3\cdot 2^l + 2\bigr)/4\bigr]$$
$$= \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1} + \frac{1}{2}\sum_{i=1}^{o}\bigl(n_j^{l-1} - 1\bigr) + \frac{1}{2}\sum_{j=o+1}^{o+e}\bigl[2^{l-1}\bigl(2^l - 3\bigr) + n_j^{l-1}\bigr],$$

which completes the proof. □

It follows that the Euler–Satake characteristics of Q associated to free and free abelian groups depend only on $\chi_{top}(Q)$ and the number and order of the cone points and corner reflectors, and in particular do not depend on how the corner reflectors are situated on the (manifold) boundary of $\mathbb{X}_Q$. In fact, this is the case more generally.

Corollary 3.4. *Suppose Q and Q' are closed, effective 2-orbifolds that have the same number and orders of cone points and corner reflectors such that $\chi_{top}(Q) = \chi_{top}(Q')$. Then $\chi_{\Gamma}^{ES}(Q) = \chi_{\Gamma}^{ES}(Q')$ for every finitely generated group Γ.*

Proof. Every finitely generated group Γ is a quotient of some free group $\mathbb{F}_l$, so that any homomorphism $\phi_x : \Gamma \to G_x$ into the isotropy group of $x \in G_0$ can be composed with the quotient map $\rho : \mathbb{F}_l \to \Gamma$ to produce a homomorphism $\phi_x \circ \rho : \mathbb{F}_l \to G_x$. It is easy to see that $\tilde{Q}_{(\phi_x)}$ and $\tilde{Q}_{(\phi_x \circ \rho)}$ are diffeomorphic; see [10, Section 3.3]. By the proof of Lemma 3.1, besides $\tilde{Q}_{(1)}$, the sectors that have nonzero Euler–Satake characteristic are determined by a homomorphism into G_x whose image has only x as a fixed-point. It follows that the number and diffeomorphism classes of these sectors depends only on the number and orders of cone points and corner reflectors, completing the proof. □

Remark 3.5. Observe in Eq. (1) that even- and odd-order cone points make the same contribution to the $\mathbb{F}_l$-Euler–Satake characteristic. To see why this is, observe that for n_j of either parity, the fixed-point set of a homomorphism $\phi_x : \mathbb{F}_l \to D_{2n_j}$ is a point whenever the image of the homomorphism is not trivial or the group generated by a single reflection. Therefore, there are $n_j(2^l - 1) + 1$ homomorphisms whose fixed-point set is not a point, and hence $2^l n_j^l - n_j(2^l - 1) - 1$ homomorphisms whose fixed point set is a point. Let T denote the set of conjugacy classes of homomorphisms whose image fix a single point, and then we have that the sum of the Euler–Satake characteristics of the $\mathbb{F}_l$-sectors associated to these homomorphisms is given by $\sum_{(\phi_x)\in T} \frac{1}{|C(\phi_x)|}$ where $C(\phi_x) = C_{D_{2n_j}}(\phi_x)$. For each ϕ_x, we have that $|C(\phi_x)||(\phi_x)| = |D_{2n_j}| = 2n_j$, and hence

$$\sum_{(\phi_x)\in T} \frac{1}{|C(\phi_x)|} = \sum_{(\phi_x)\in T} \frac{|(\phi_x)|}{|D_{2n_j}|} = \frac{1}{|D_{2n_j}|}\sum_{(\phi_x)\in T}\bigl|(\phi_x)\bigr|$$
$$= \frac{2^l n_j^l - n_j(2^l - 1) - 1}{2n_j}.$$

This, along with [23, Eq. (13.3.4)] and [7, Proposition 3.1], yields an alternative proof of Eq. (1) that does not require the counting in Lemma 3.1. However, Lemmas 3.1 and 3.2 are required to obtain Eq. (2).

In exactly the same way, one can use Lemmas 3.1 and 3.2 to compute the $\mathbb{Z}^l$- and $\mathbb{F}_l$-extensions of the usual Euler characteristic and Betti numbers. We state the results of these computations.

Corollary 3.6 *(Γ-Betti numbers). Let Q be a closed, effective 2-orbifold with singularities denoted as in Lemma 3.1. Then for each integer $l \geqslant 0$, the zeroth $\mathbb{F}_l$-Betti number is given by*

$$b^0_{\mathbb{F}_l}(Q) = b_0(Q) + d\left(2^l - 1\right) - k + \sum_{i=1}^{k} m_i^l + \frac{1}{2}\sum_{j=1}^{o}\left[\left(n_j^{l-1} - 1\right)\left(2^l - 1\right) + n_j^l - 1\right]$$
$$+ \frac{1}{2}\sum_{j=o+1}^{o+e}\left[2^l\left(2^l - 2\right) + 2\left(2^l - 1\right)n_j^{l-1} + n_j^l\right],$$

the first is given by

$$b^1_{\mathbb{F}_l}(Q) = b_1(Q) + d\left(2^l - 1\right),$$

and $b^2_{\mathbb{F}_l}(Q) = b_2(Q)$. Similarly, the zeroth $\mathbb{Z}^l$-Betti number is given by

$$b^0_{\mathbb{Z}^l}(Q) = b_0(Q) + d\left(2^l - 1\right) - k + \sum_{i=1}^{k} m_i^l + \frac{1}{2}\sum_{j=1}^{o}\left(n_j^{l-1} - 1\right) + \frac{1}{2}\sum_{j=o+1}^{o+e}\left[2^l\left(2^{l+1} - 3\right) + n_j^l\right],$$

the first is given by

$$b^1_{\mathbb{Z}^l}(Q) = b_1(Q) + d\left(2^l - 1\right),$$

and $b^2_{\mathbb{Z}^l}(Q) = b_2(Q)$.

Note in particular that the Betti numbers depend on the number d of (manifold) boundary components with no even-order corner reflectors. Hence a result analogous to Corollary 3.4 does not hold for these invariants. This is not the case for the extensions of the Euler characteristic, as follows from the following.

Corollary 3.7 *(Γ-Euler characteristics). Let Q be a closed, effective 2-orbifold with singularities denoted as in Lemma 3.1. Then for each integer $l \geqslant 0$,*

$$\chi^{top}_{\mathbb{F}_l}(Q) = \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^l + \frac{1}{2}\sum_{j=1}^{o}\left[\left(n_j^{l-1} - 1\right)\left(2^l - 1\right) + n_j^l - 1\right]$$
$$+ \frac{1}{2}\sum_{j=o+1}^{o+e}\left[2^l\left(2^l - 2\right) + 2\left(2^l - 1\right)n_j^{l-1} + n_j^l\right], \tag{3}$$

and

$$\chi^{top}_{\mathbb{Z}^l}(Q) = \chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^l + \frac{1}{2}\sum_{j=1}^{o}\left(n_j^l - 1\right) + \frac{1}{2}\sum_{j=o+1}^{o+e}\left[2^l\left(2^{l+1} - 3\right) + n_j^l\right]. \tag{4}$$

Of particular interest is Eq. (4), from which it is evident that

$$\chi^{top}_{\mathbb{Z}^l}(Q) = \chi^{ES}_{\mathbb{Z}^{l+1}}(Q)$$

in this case. In fact, this is the case for any closed orbifold; see [20, Theorem 3.2] and [10, Eq. (4.5)].

4. Determining singularities from Γ-Euler–Satake characteristics

4.1. Proof of Theorem 1.1 and its consequences

In this subsection, we determine the extent to which the Γ-Euler–Satake characteristics associated to free and free abelian groups determine the singularities of closed, connected, effective 2-orbifolds. This collection of Euler–Satake characteristics determines the number and order of cone points and corner reflectors of every order with one exception. It does not distinguish between a cone point of order 4 and the pair of a cone point of order 2 and a corner reflector of order 2. Hence, the hypothesis in Theorem 1.1 that neither Q nor Q' have cone points of order 4 is necessary. We illustrate this with the following.

Example 4.1. Let S be any surface with boundary. Let Q have underlying space S with a reflector line at the (manifold) boundary, k cone points of order 4, and no corner reflectors. Similarly, let Q' have underlying space S with a reflector line at the (manifold) boundary, k cone points of order 2, and k corner reflectors of order 2. Then Eqs. (1) and (2) yield

$$\chi_\Gamma^{ES}(Q) = \chi_\Gamma^{ES}(Q') = \chi_{top}(S) - k + k \cdot 4^{l-1}$$

for any free or free abelian Γ of rank l.

The singularities in the above example, however, are the only singularities between which the collection of free and free abelian Euler–Satake characteristics cannot distinguish.

Proposition 4.2. *Suppose Q and Q' are closed, connected, effective 2-orbifolds such that $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for infinitely many integers l and $\chi_{\mathbb{F}_\lambda}^{ES}(Q) = \chi_{\mathbb{F}_\lambda}^{ES}(Q')$ for infinitely many integers λ. Then $\chi_{top}(Q) = \chi_{top}(Q')$, and the number and order of cone points and corner reflectors of Q coincide with those of Q' up to exchanging cone points of order 4 each with a cone point of order 2 and a corner reflector of order 2.*

Proof. Suppose Q and Q' are distinct, closed, connected, effective 2-orbifolds such that there is an infinite collection $\mathfrak{L}_1$ of nonnegative integers such that $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for each $l \in \mathfrak{L}_1$, and such that there is an infinite collection $\mathfrak{L}_2$ of nonnegative integers such that $\chi_{\mathbb{F}_\lambda}^{ES}(Q) = \chi_{\mathbb{F}_\lambda}^{ES}(Q')$ for each $\lambda \in \mathfrak{L}_2$. Assume Q has k cone points of orders $m_1, m_2, \dots, m_k$, has o corner reflectors of odd orders $n_1, n_2, \dots, n_o$, and has e corner reflectors of even orders $n_{o+1}, n_{o+2}, \dots, n_{o+e}$. Similarly, assume Q' has κ cone points of orders $\mu_1, \mu_2, \dots, \mu_\kappa$, has σ corner reflectors of odd orders $\nu_1, \nu_2, \dots, \nu_\sigma$, and has ε corner reflectors of even orders $\nu_{\sigma+1}, \nu_{\sigma+2}, \dots, \nu_{\sigma+\varepsilon}$. It follows from Corollary 3.3 that if $m_i = \mu_\alpha$ for some i and α, then the orbifolds formed by replacing a neighborhood of each of these cone points in Q and Q' with a nonsingular disk still satisfy the hypotheses above; see [7, Lemma 3.5]. The same argument holds when corner reflectors of the same order are replaced by reflector lines, and so we may assume without loss of generality that Q and Q' do not have cone points or corner reflectors of the same order, i.e. that $m_i \neq \mu_\alpha$ for $1 \leqslant i \leqslant k$ and $1 \leqslant \alpha \leqslant \kappa$, and $n_j \neq \nu_\beta$ for $1 \leqslant j \leqslant o+e$ and $1 \leqslant \beta \leqslant \sigma + \varepsilon$. Note that neither orbifold need have singularities of each type.

Applying Eq. (1), we have that for each $\lambda \in \mathfrak{L}_2$,

$$\chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{\lambda-1} + 2^{\lambda-1} \sum_{j=1}^{o+e} \bigl(n_j^{\lambda-1} - 1\bigr) = \chi_{top}(Q') - \kappa + \sum_{\alpha=1}^{\kappa} \mu_\alpha^{\lambda-1} + 2^{\lambda-1} \sum_{\beta=1}^{\sigma+\epsilon} \bigl(\nu_\beta^{\lambda-1} - 1\bigr).$$

Evidently, if either Q or Q' is nonsingular, then the result is trivial, so assume not. As each $m_i, n_j, \mu_\alpha, \nu_\beta \geqslant 2$, both sides of this equation strictly increases with λ and hence are positive for sufficiently large λ. Then we have

$$\frac{\chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{\lambda-1} + \sum_{j=1}^{o+e} [(2n_j)^{\lambda-1} - 2^{\lambda-1}]}{\chi_{top}(Q') - \kappa + \sum_{\alpha=1}^{\kappa} \mu_\alpha^{\lambda-1} + \sum_{\beta=1}^{\sigma+\epsilon} [(2\nu_\beta)^{\lambda-1} - 2^{\lambda-1}]} = 1.$$

Considering the limit as $\lambda \to \infty$ and noting that $2n_j > 2$ for each j and $2\nu_\beta > 2$ for each β, it is easy to see that $\max_{i,j}\{m_i, 2n_j\} = \max_{\alpha,\beta}\{\mu_\alpha, 2\nu_\beta\}$.

We consider two cases. For the first case, suppose $\max_{i,j}\{m_i, 2n_j\} = m_I$ for some I. Then as Q' has no cone points of order m_I by hypothesis, it follows that $m_I > \mu_\alpha$ for each α, and hence that $m_I = 2\nu_B$ for some B. In particular, this implies that $m_I \geqslant 4$. Then as $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for each $l \in \mathfrak{L}_1$, we have by Eq. (2) that for l sufficiently large,

$$\frac{\chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1} + \frac{1}{2}\sum_{j=1}^{o} (n_j^{l-1} - 1) + \frac{1}{2}\sum_{j=o+1}^{o+e} [2 \cdot 4^{l-1} - 3 \cdot 2^{l-1} + n_j^{l-1}]}{\chi_{top}(Q') - \kappa + \sum_{\alpha=1}^{\kappa} \mu_\alpha^{l-1} + \frac{1}{2}\sum_{\beta=1}^{\sigma} (\nu_\beta^{l-1} - 1) + \frac{1}{2}\sum_{\beta=\sigma+1}^{\sigma+\varepsilon} [2 \cdot 4^{l-1} - 3 \cdot 2^{l-1} + \nu_\beta^{l-1}]} = 1.$$

Now, m_I is strictly greater than each n_j, μ_α, and ν_β. So again considering the limit as $l \to \infty$, we see that this equation can only be satisfied for infinitely many l if $m_I = 4$. This implies that $n_j \leqslant 2$ and $\nu_\beta \leqslant 2$ for each j, β, so that neither Q nor Q' have odd-order corner reflectors. Moreover, Q' must have at least one corner reflector of order 2, implying by hypothesis that Q has no corner reflectors at all. Therefore,

$$\frac{\chi_{top}(Q) - k + \sum_{i=1}^{k} m_i^{l-1}}{\chi_{top}(Q') - \kappa + \sum_{\alpha=1}^{\kappa} \mu_\alpha^{l-1} + \varepsilon[4^{l-1} - 2^{l-1}]} = 1.$$

Considering the limit, it follows that Q has exactly ε cone points of order 4, and hence we have

$$\frac{\chi_{top}(Q) - k + \sum_{m_i=2,3} m_i^{l-1}}{\chi_{top}(Q') - \kappa + \sum_{\alpha=1}^{\kappa} \mu_\alpha^{l-1} - \varepsilon 2^{l-1}} = 1$$

where each $\mu_\alpha < 4$. With this, noting that no m_i can be equal to any μ_α, it is easy to see that Q has no cone points of orders 2 or 3, Q' has exactly ε cone points of order 2, $\kappa = k = \varepsilon$, and $\chi_{top}(Q) = \chi_{top}(Q')$.

For the second case, suppose $\max_{i,j}\{m_i, 2n_j\} = 2n_J$ for some J. Then as Q' can have no corner reflectors of order n_J by hypothesis, we have that $2n_J = \max_{\alpha,\beta}\{\mu_\alpha, 2\nu_\beta\} = \mu_A$ for some A. Hence switching the roles of Q and Q' in the above argument yields the same conclusion, completing the proof. $\square$

Note in particular that Theorem 1.1 follows from Proposition 4.2. As well, it follows that Q and Q' have the same Γ-Euler–Satake characteristics for free and free abelian Γ of arbitrarily large rank if and only if they have the same Γ-Euler–Satake characteristics for every free and free abelian Γ. If additionally Q and Q' do not have any cone points of order 4, Proposition 4.2 and Corollary 3.4 imply that $\chi_\Gamma^{ES}(Q) = \chi_\Gamma^{ES}(Q')$ for every finitely generated group Γ. Hence, the class of closed, connected, effective, 2-orbifolds with no cone points of order 4 is an appropriate class in which to test the free and free abelian Euler–Satake characteristics against smaller collections of invariants. We also have the following.

Corollary 4.3. *Let Q be a closed, connected, effective 2-orbifold. Then there is a finite number of diffeomorphism classes of closed, effective, connected orbifolds Q' such that $\chi_\Gamma^{ES}(Q) = \chi_\Gamma^{ES}(Q')$ for every free and free abelian Γ.*

Proof. If $\chi_\Gamma^{ES}(Q) = \chi_\Gamma^{ES}(Q')$ for every free and free abelian Γ, then $\chi_{top}(Q') = \chi_{top}(Q)$ by Proposition 4.2. The underlying space of each Q' is determined by whether it is orientable, its genus g, and the number d of connected components of its (manifold) boundary; we have $\chi_{top}(Q') = 2 - 2g - d$ if Q' is orientable and $\chi_{top}(Q') = 2 - g - d$ otherwise; see e.g. [13]. Given $\chi_{top}(Q)$, there are a finite number of nonnegative integer solutions of

$$d + 2g = 2 - \chi_{top}(Q)$$

and

$$d + g = 2 - \chi_{top}(Q)$$

for g and d, and hence a finite number of possibilities for the underlying space of Q.

It also follows from Proposition 4.2 that as Q has a finite number of cone points of orders 2 and 4, there are a finite number of possibilities of the number and orders of cone points and corner reflectors of Q'. Given a choice of the underlying space of Q' and a choice of the number and orders of cone points and corner reflectors, there is a finite number of ways of situating the corner reflectors on the (manifold) boundary, up to diffeomorphism. It follows that there are only finitely many diffeomorphism classes of Q', completing the proof. $\square$

4.2. Proof of Theorem 1.2

In this subsection, we prove Theorem 1.2. In particular, given an integer $L \geqslant 0$, we construct orbifolds Q, Q', $\mathcal{Q}$, and $\mathcal{Q}'$ as required by the theorem.

Proof. In [7, Lemma 3.11], it was shown that for any $L \geqslant 2$ and $g \geqslant 0$, there are effective, orientable, closed 2-orbifolds, both with underlying space a surface of genus g and the same number k of cone points, whose $\mathbb{Z}^l$-Euler–Satake characteristics coincide for $l \leqslant L$. If R is any collection of 2^{L-2} integers, then the orders of the cone points of these orbifolds can be taken to be in the set $\{2q+1, 2q^2+q, q+2, 2q+q^2 \mid q \in R\}$. Letting $2 \leqslant n_1 \leqslant \cdots \leqslant n_k$ and $2 \leqslant \nu_1 \leqslant \cdots \leqslant \nu_k$ denote the orders of the cone points of these two orbifolds, respectively, it follows that

$$\sum_{j=1}^{k} n_j^{l-1} = \sum_{\alpha=1}^{k} \nu_\alpha^{l-1}$$

for each $l \leqslant L$. Moreover, each n_i and ν_α are elements of $\{2q+1, 2q^2+q, q+2, 2q+q^2 \mid q \in R\}$, and lists $n_1, \ldots, n_k$ and $\nu_1, \ldots, \nu_k$ do not coincide.

So let $L \geqslant 2$ and pick $n_1, \ldots, n_k$ and $\nu_1, \ldots, \nu_k$ as above. By letting $R = \{3, 5, \ldots, 2^{L-1}+1\}$, we can ensure that each n_j and ν_α is odd. If $n_j = \nu_\alpha$ for some j and α, then we can remove these to produce a similar set of integers, so we can assume this not the case.

Let S be any surface with boundary. Following the notation in [23], we let $Q = S(2\nu_1, \ldots, 2\nu_k; n_1, \ldots, n_k)$ denote an orbifold with underlying space S, k cone points of orders $2\nu_1, \ldots, 2\nu_k$, and k corner reflectors of orders $n_1, \ldots, n_k$. Let $Q' = S(2n_1, \ldots, 2n_k; \nu_1, \ldots, \nu_k)$ be an orbifold with underlying space S, k cone points of orders $2n_1, \ldots, 2n_k$, and k corner reflectors of orders $\nu_1, \ldots, \nu_k$. Then a direct computation applying Eqs. (1) and (2) yields $\chi_{\mathbb{F}_\lambda}^{ES}(Q) = \chi_{\mathbb{F}_\lambda}^{ES}(Q')$ for each $\lambda \geqslant 0$, and $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for each $l \leqslant L$. It follows that $\chi_{\mathbb{F}_\lambda}^{ES}(Q) = \chi_{\mathbb{F}_\lambda}^{ES}(Q')$ for each $\lambda \geqslant 0$ and $\chi_{\mathbb{Z}^l}^{ES}(Q) = \chi_{\mathbb{Z}^l}^{ES}(Q')$ for each $l \leqslant L$. Moreover, as $n_j \neq \nu_\alpha$ for each j and α, Q and Q' do not have any cone points or corner reflectors of the same order.

Similarly, let $\mathcal{Q} = S(\nu_1, \ldots, \nu_k; n_1, n_1, n_2, n_2, \ldots, n_k, n_k)$ be an orbifold with underlying space S, k cone points of orders $\nu_1, \ldots, \nu_k$, and $2k$ corner reflectors, two of order n_j for each j. Let $\mathcal{Q}' = S(n_1, \ldots, n_k; \nu_1, \nu_1, \nu_2, \nu_2, \ldots, \nu_k, \nu_k)$ be an orbifold with underlying space S, k cone points of orders $n_1,\dots,n_k$, and $2k$ corner reflectors, two of order ν_α for each α. Then a direct computation yields $\chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}) = \chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}')$ for any $l \geqslant 0$, and $\chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}) = \chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}')$ for each $\lambda \leqslant L$. Therefore, $\chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}) = \chi^{ES}_{\mathbb{Z}^l}(\mathcal{Q}')$ for each $l \geqslant 0$, and $\chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}) = \chi^{ES}_{\mathbb{F}_\lambda}(\mathcal{Q}')$ for each $\lambda \leqslant L$. As well, $\mathcal{Q}$ and $\mathcal{Q}'$ do not have any cone points or corner reflectors of the same order, completing the proof. $\square$

It follows that Theorem 1.1 cannot be obtained by considering Euler–Satake characteristics associated to a smaller class of free or free abelian groups.

4.3. Determining singularities of 2-orbifolds

We end with a brief observation regarding collections of Γ-Euler–Satake characteristics that determine as much information about closed, connected, effective 2-orbifolds as possible.

Corollary 4.4. *Suppose Q and Q' are closed, connected effective 2-orbifolds such that $\chi^{ES}_{\mathbb{Z}^l}(Q) = \chi^{ES}_{\mathbb{Z}^l}(Q')$ for infinitely many integers l, $\chi^{ES}_{\mathbb{F}_\lambda}(Q) = \chi^{ES}_{\mathbb{F}_\lambda}(Q')$ for infinitely many integers λ, and $\chi^{ES}_{\mathbb{Z}/2\mathbb{Z}}(Q) = \chi^{ES}_{\mathbb{Z}/2\mathbb{Z}}(Q')$. Then $\chi_{top}(Q) = \chi_{top}(Q')$, and Q and Q' have the same number of cone points and corner reflectors of each order.*

Proof. As in the proof of Theorem 1.1, we assume without loss of generality that Q and Q' have no cone points or corner reflectors of the same order. Then as Q and Q' satisfy the hypotheses of Theorem 1.1, we can assume that $\chi_{top}(Q) = \chi_{top}(Q')$, Q has k cone points of order 4 and no corner reflectors, and Q' has k cone points of order 2 and k corner reflectors of order 2. By [23, Eq. (13.3.4)], we have that

$$\chi_{ES}(\tilde{Q}_{(1)}) = \chi_{ES}(Q) = \chi_{top}(Q) - \frac{3k}{4}.$$

It is easy to see that the $\mathbb{Z}/2\mathbb{Z}$-sectors of Q consist of $\tilde{Q}_{(1)}$ along with k sectors of the form $\mathbb{Z}/4\mathbb{Z} \ltimes_{triv} \{pt\}$. Hence,

$$\chi^{ES}_{\mathbb{Z}/2\mathbb{Z}}(Q) = \chi_{top}(Q) - \frac{3k}{4} + \frac{k}{4} = \chi_{top}(Q) - \frac{k}{2}.$$

Similarly,

$$\chi_{ES}\big(\tilde{Q}'_{(1)}\big) = \chi_{ES}\big(Q'\big) = \chi_{top}(Q) - \frac{3k}{4}.$$

The $\mathbb{Z}/2\mathbb{Z}$-sectors of Q' with nonzero Euler–Satake characteristic consist of $\tilde{Q}'_{(1)}$, k sectors of the form $\mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$ corresponding to the nontrivial homomorphisms into isotropy groups of cone points, and k sectors of the form $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ltimes_{triv} \{pt\}$ corresponding to homomorphisms whose image contains the nontrivial rotation in the isotropy group of a corner reflector. Therefore,

$$\chi^{ES}_{\mathbb{Z}/2\mathbb{Z}}\big(Q'\big) = \chi_{top}\big(Q'\big) - \frac{3k}{4} + \frac{k}{2} + \frac{k}{4} = \chi_{top}\big(Q'\big).$$

It follows that $k = 0$. $\square$

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